A spectral identity on Jacobi polynomials and its analytic implications

Article (Accepted Version)

A spectral identity on Jacobi polynomials and its analytic implications

Richard Olu Awonusika Ali Taheri
Department of Mathematics, University of Sussex, Brighton, UK.
R.Awonusika@sussex.ac.uk A.Taheri@sussex.ac.uk

Abstract

The Jacobi coefficients $c^{\ell}_{j}(\alpha,\beta)$ $(1 \le j \le \ell,\ \alpha,\beta > -1)$ are linked to the Maclaurin spectral expansion of the Schwartz kernel of functions of the Laplacian on a compact rank one symmetric space. It is proved that these coefficients can be computed by transforming the even derivatives of the the Jacobi polynomials $P^{(\alpha,\beta)}_{k}$ $(k \ge 0, \alpha, \beta > -1)$ into a spectral sum associated with the Jacobi operator. The first few coefficients are explicitly computed and a direct trace interpretation of the Maclaurin coefficients is presented.

Keywords: Jacobi coefficients, Maclaurin spectral functions, Symmetric spaces, Laplace-Beltrami operator, Schwartz kernel, Jacobi polynomials.
MSC (2000): 33C05, 33C45, 35A08, 35C05, 35C10, 35C15.

1 Introduction

The Jacobi polynomials have a close connection with the Laplace operator on compact rank one symmetric spaces. They represent the spherical functions on these spaces and serve as the key ingredient in describing the spectral projections and the spectral measure associated with the Laplacian. In this note we bring this connection to the fore by showing that a set of spectral and geometric quantities associated with Jacobi operator fully describe the Maclaurin coefficients relating to the Schwartz kernel of operators in the functional calculus of the Laplacian.

The Jacobi polynomials $P_k^{(\alpha,\beta)}$ (integer $k \geq 0$, real $\alpha, \beta > -1$) constitute an orthogonal family of polynomials with the generating function relation [1]

$$\frac{1}{(1-z+R)^{\alpha}(1+z+R)^{\beta}} = R2^{-(\alpha+\beta)} \sum_{k=0}^{\infty} P_k^{(\alpha,\beta)}(t) z^k, \qquad |z| < 1, \quad (1.1)$$

where $R = \sqrt{1-2tz+z^2}$. It is seen that $P_k^{(\alpha,\beta)}$ is a degree k polynomial admitting the truncated series representation

$$P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+\beta+k+1)} \sum_{l=0}^{k} \binom{k}{l} \frac{\Gamma(\alpha+\beta+k+l+1)}{2^l \Gamma(\alpha+l+1) k!} (t-1)^l, \quad (1.2)$$

and satisfying the classical Rodrigues' representation formula

$$P_k^{(\alpha,\beta)}(t) = \frac{(-1)^k}{2^k k!} (1-t)^{-\alpha} (1+t)^{-\beta} \frac{d^k}{dt^k} \left[(1-t)^{k+\alpha} (1+t)^{k+\beta} \right]. \quad (1.3)$$

The Jacobi polynomial $y = P_k^{(\alpha,\beta)}$ satisfies the second-order homogenous linear differential equation – the Jacobi equation:

$$\left(1-t^2\right) \frac{d^2 y}{dt^2} - (\alpha-\beta+(\alpha+\beta+2)t) \frac{dy}{dt} + k(k+\alpha+\beta+1) y = 0, \quad (1.4)$$

that in turn constitute a regular Sturm-Liouville system with the associated Jacobi operator a positive selfadjoint second order linear differential operator in the weighted space $L^2[-1,1;(1-t)^{\alpha}(1+t)^{\beta}dt]$. The spectrum here is discrete and given by the sequence of eigenvalues and eigenfunctions

$$\lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1), \qquad y = P_k^{(\alpha,\beta)}(t), \qquad k \geq 0. \qquad (1.5)$$

In particular and as a result the Jacobi polynomials satisfy the orthogonality relations

$$\begin{aligned} \mathsf{c}^{(\alpha,\beta)} \delta_{k,m} &= \langle P_k^{(\alpha,\beta)}, P_m^{(\alpha,\beta)} \rangle_{L^2[-1,1;(1-t)^{\alpha}(1+t)^{\beta}dt]} \qquad k, m \geq 0 \\ &= \int_{-1}^{1} P_k^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) \, (1-t)^{\alpha} \, (1+t)^{\beta} \, dt. \qquad (1.6) \end{aligned}$$

Here $\delta_{k,m}$ is the usual Kronecker delta and the constants $\mathsf{c}^{(\alpha,\beta)}$ on the left (that are of relevance only when $k = m$) are given by

$$\mathsf{c}^{(\alpha,\beta)} = 2^{\alpha+\beta+1} \frac{(\alpha+1)_k (\beta+1)_k (\alpha+\beta+k+1)}{k!(\alpha+\beta+2)_k (\beta+\alpha+2k+1)} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \quad (1.7)$$

[1] For more information the interested reader is referred to [5, 8, 9] and [16, 17].

Here and below $(x)_k = \Gamma(x+k)/\Gamma(x)$ denotes the rising factorial. Now for $m \geq 1$ the Jacobi polynomials satisfy the differential recursion formula

$$\frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(k+m+\alpha+\beta+1)}{2^m \Gamma(k+\alpha+\beta+1)} P_{k-m}^{(\alpha+m,\beta+m)}(t), \tag{1.8}$$

along with the reflection symmetry and pointwise identities

$$P_k^{(\alpha,\beta)}(-t) = (-1)^k P_k^{(\beta,\alpha)}(t), \quad P_k^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_k}{k!}. \tag{1.9}$$

The Jacobi, Gegenbauer and Legendre polynomials are related to one-another for suitable choice of (α, β) parameters as given by (with $k \geq 0$, $\nu > -1/2$)

$$P_k^{(0,0)}(t) = C_k^{1/2}(t) = P_k(t), \qquad C_k^{\nu}(t) = \frac{(2\nu)_k}{(\nu+1/2)_k} P_k^{(\nu-1/2,\nu-1/2)}(t).$$

Having all this in place we note that in sequel it is often more advantageous to use the normalised form of the Jacobi polynomials, written $\mathscr{P}_k^{(\alpha,\beta)}$, and defined by

$$\mathscr{P}_k^{(\alpha,\beta)}(t) = \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\alpha,\beta)}(1)} = \frac{k!}{(\alpha+1)_k} P_k^{(\alpha,\beta)}(t). \tag{1.10}$$

Note that as a result of this normalisation $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$.

2 Jacobi coefficients and a spectral identity

We start by proving a differential-spectral identity relating the derivatives of the Jacobi polynomial to a suitable weighted sum of the integer powers of the eigenvalues of the Jacobi operator. Applications and implications of this will be discussed later on.

Theorem 2.1. (*Jacobi coefficients*) *The normalised Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}$ with $k \geq 0$, $\alpha, \beta > -1$ satisfies the differential-spectral identity*

$$\frac{d^{2\ell}}{d\theta^{2\ell}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} = \sum_{j=1}^{\ell} c_j^{\ell}(\alpha,\beta) [\lambda_k^{(\alpha,\beta)}]^j = \mathscr{R}_\ell(\lambda_k^{(\alpha,\beta)}). \tag{2.1}$$

Here $(c_j^\ell(\alpha,\beta) : 1 \leq j \leq \ell)$ are suitable constants, $\lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1)$ with $k \geq 0$ are the eigenvalues of the Jacobi operator from (1.4)-(1.5) *and $\mathscr{R}_\ell = \mathscr{R}_\ell(X)$ is the degree ℓ polynomial defined via* (2.1) [*see also* (2.9)].

Proof. Basic considerations shows that for suitable scalars A^ℓ_m $(1 \le m \le \ell)$ and upon invoking the differential-recursion formula (1.8) we can write

$$\begin{aligned}\frac{d^{2\ell}}{d\theta^{2\ell}} P_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} &= \sum_{m=1}^{\ell} \mathsf{A}^\ell_m \frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t)\Big|_{t=1} \qquad (2.2)\\ &= \sum_{m=1}^{\ell} \frac{\mathsf{A}^\ell_m \Gamma(k+m+\alpha+\beta+1)}{2^m \Gamma(k+\alpha+\beta+1)} P_{k-m}^{(\alpha+m,\beta+m)}(1)\\ &= \sum_{m=1}^{\ell} \frac{\mathsf{A}^\ell_m \Gamma(k+\alpha+1)}{\Gamma(\alpha+m+1)} \frac{2^{-m}\Gamma(k+\alpha+\beta+m+1)}{\Gamma(k+\alpha+\beta+1)(k-m)!}\\ &= \sum_{m=1}^{\ell} \frac{\mathsf{A}^\ell_m \Gamma(\alpha+1) 2^{-m} \Gamma(k+\alpha+\beta+m+1)k!}{\Gamma(k+\alpha+\beta+1)\Gamma(\alpha+m+1)(k-m)!} P_k^{(\alpha,\beta)}(1).\end{aligned}$$

Now anticipating on the last line above, using the basic properties of the Gamma function and a subsequent expansion, it follows that for suitable choice of constants $\mathsf{B}^m_j = \mathsf{B}^m_j(\alpha,\beta)$ we can write

$$\begin{aligned}\frac{\Gamma(k+\alpha+\beta+m+1)k!}{\Gamma(k+\alpha+\beta+1)(k-m)!} &= \prod_{p=0}^{m-1}(k+\alpha+\beta+p+1) \times \prod_{p=0}^{m-1}(k-p)\\ &= \prod_{p=0}^{m-1}\Big(k(k+\alpha+\beta+1) - p(p+\alpha+\beta+1)\Big)\\ &= \sum_{j=1}^{m} \mathsf{B}^m_j [k(k+\alpha+\beta+1)]^m = \sum_{j=1}^{m} \mathsf{B}^m_j [\lambda_k^{(\alpha,\beta)}]^j\end{aligned}$$

where the penultimate identity follows from a straightforward induction on m. Substituting back into (2.2) gives the required conclusion. $\square$

Now let $-\Delta_{\mathscr{X}}$ denote the (*positive*) Laplace operator on a compact rank one symmetric space $\mathscr{X}$. Then by basic spectral theory upon taking Φ in the functional calculus of $-\Delta_{\mathscr{X}}$ the Schwarz kernel of $\Phi(-\Delta_{\mathscr{X}})$ can be expressed by the spectral sum

$$K_\Phi(x,y) = \sum_{k=0}^{\infty} \frac{M_k \Phi(\lambda_k^{(\alpha,\beta)})}{\mathrm{Vol}(\mathscr{X})} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta). \qquad (2.3)$$

Here θ is the geodesic distance between x, y, $\mathrm{Vol}(\mathscr{X})$ is the volume of $\mathscr{X}$ and $\lambda_k(\mathscr{X}) = \lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1)$ are the numerically distinct eigenvalues of $-\Delta_{\mathscr{X}}$ with multiplicities $M_k = M_k(\mathscr{X})$ where $k \ge 0$. (See below for more.)

Before proceeding further let us note that the compact rank one symmetric spaces that are of particular interest are the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbf{P}^n(\mathbb{R}) = \mathbb{S}^n/\{\pm\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbf{P}^n(\mathbb{C}) = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$ (of real dimension $2n$), the quaternionic projective space $\mathbf{P}^n(\mathbb{H}) = \mathbf{Sp}(n+1)/\mathbf{Sp}(n)\times\mathbf{Sp}(1)$ (of real dimension $4n$) and the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$ (of real dimension 16). [2]

For the sake of future reference we now present some of the main spectral geometric quantities associated with the above spaces. The formulation of these, in the simply-connected case are given, in turn, by the eigenvalues (with $k \geq 0$ and $\varrho = (a + b/2)/2$ see Table 1 below)

$$\lambda_k(\mathscr{X}) = (\varrho + k)^2 - \varrho^2 = k(k + a + b/2); \tag{2.4}$$

the radial part of the Laplacian $\partial_\theta^2 + (a\cot\theta + (1/2)b\cot(\theta/2))\,\partial_\theta$; the multiplicity of the eigenvalue $\lambda_k(\mathscr{X})$ (with $k \geq 0$ and $N = a + b + 1$) given by

$$M_k(\mathscr{X}) = \frac{2(k+\varrho)\Gamma(k+2\varrho)\Gamma\left((a+1)/2\right)\Gamma\left(k+N/2\right)}{k!\Gamma(2\varrho+1)\Gamma\left(N/2\right)\Gamma\left(k+(a+1)/2\right)}, \tag{2.5}$$

and the volume

$$\mathrm{Vol}(\mathscr{X}) = \pi^{\frac{N}{2}} \frac{2^N\Gamma\left((a+1)/2\right)}{\Gamma\left((N+a+1)/2\right)}. \tag{2.6}$$

In the non simply-connected case $\mathbf{P}^n(\mathbb{R})$ the counterparts of these quantities are obtained upon using standard arguments from those of its double cover $\mathbb{S}^n$. In Table 1 below we gather together the values of the parameters a, b, N, α, β and ϱ for the symmetric space $\mathscr{X}$. Note that here N is the real dimension of $\mathscr{X}$, $\alpha = (N-2)/2$ and $\beta = (a-1)/2$.

Table 1: The Parameters a, b, N, α, β and ϱ associated with $\mathscr{X}$

$\mathscr{X}$	N	a	b	α	β	ϱ
$\mathbb{S}^n$	n	$n-1$	0	$(n-2)/2$	$(n-2)/2$	$(n-1)/2$
$\mathbf{P}^n(\mathbb{R})$	n	$n-1$	0	$(n-2)/2$	$(n-2)/2$	$(n-1)/2$
$\mathbf{P}^n(\mathbb{C})$	$2n$	1	$2(n-1)$	$n-1$	0	$n/2$
$\mathbf{P}^n(\mathbb{H})$	$4n$	3	$4(n-1)$	$2n-1$	1	$n+1/2$
$\mathbf{P}^2(\mathrm{Cay})$	16	7	8	7	3	$11/2$

[2]With the exception of the sphere $\mathbb{S}^1$ and the real projective space $\mathbf{P}^n(\mathbb{R})$ (with $n \geq 1$) all these spaces are simply-connected. Indeed $\pi_1(\mathbb{S}^1) \cong \pi_1(\mathbf{P}^1(\mathbb{R})) \cong \mathbb{Z}$ whilst $\pi_1(\mathbf{P}^n(\mathbb{R})) \cong \mathbb{Z}_2$ for $n \geq 2$.

Tables 2-3 below illustrate some of the main spectral geometric quantities [*cf.* (2.4)-(2.6)] associated with the symmetric spaces described above. For further reference and discussion see also [1], [4, 6, 7] and [17, 18].

Table 2: Data for the symmetric spaces $\mathbb{S}^n$, $\mathbf{P}^n(\mathbb{R})$ and $\mathbf{P}^n(\mathbb{C})$

$\mathscr{X}$	$\mathbb{S}^n$	$\mathbf{P}^n(\mathbb{R})$	$\mathbf{P}^n(\mathbb{C})$
$\lambda_k(\mathscr{X})$	$k(k+n-1)$	$2k(2k+n-1)$	$k(k+n)$
$M_k(\mathscr{X})$	$(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$	$(4k+n-1)\frac{(2k+n-2)!}{(2k)!(n-1)!}$	$\frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2$
$\mathrm{Vol}(\mathscr{X})$	$\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$	$\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$	$\frac{4^n\pi^n}{n!}$

Now as by (2.3) K_Φ is an even function of the geodesic distance its Maclaurin expansion about $\theta = 0$ takes the form

$$K_\Phi = \sum_{\ell=0}^{\infty} \frac{\theta^{2\ell}}{(2\ell)!} \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} K_\Phi \Big|_{\theta=0} = \sum_{\ell=0}^{\infty} b_{2\ell}^n \frac{\theta^{2\ell}}{(2\ell)!} \tag{2.7}$$

with $b_{2\ell}^n = b_{2\ell}^n[\Phi]$ ($\ell \geq 0$) the associated Maclaurin coefficients. A direct calculation upon invoking (2.1) now shows that

$$\begin{aligned} b_{2\ell}^n = \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} K_\Phi \Big|_{\theta=0} &= \sum_{k=0}^{\infty} \frac{M_k \Phi(\lambda_k^{(\alpha,\beta)})}{\mathrm{Vol}(\mathscr{X})} \frac{d^{2\ell}}{d\theta^{2\ell}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) \Big|_{\theta=0} \\ &= \sum_{k=0}^{\infty} \frac{M_k \Phi(\lambda_k^{(\alpha,\beta)})}{\mathrm{Vol}(\mathscr{X})} \sum_{j=1}^{\ell} c_j^\ell(\alpha,\beta)[\lambda_k^{(\alpha,\beta)}]^j \\ &= \frac{1}{\mathrm{Vol}(\mathscr{X})} \mathrm{tr}\,[\mathscr{R}_\ell \Phi](-\Delta_{\mathscr{X}}) \end{aligned} \tag{2.8}$$

where $\mathscr{R}_\ell = \mathscr{R}_\ell(X)$ is the polynomial of degree ℓ built out of the Jacobi coefficients $(c_j^\ell(\alpha,\beta) : 1 \leq j \leq \ell)$, specifically,

$$\mathscr{R}_\ell(X) = \sum_{j=1}^{\ell} c_j^\ell(\alpha,\beta) X^j. \tag{2.9}$$

Note in particular that in the case of the heat semigroup with $\Phi_t(X) = e^{-tX}$ the Maclaurin coefficients of the heat kernel $K_t(x,y) := K_{\Phi_t}(x,y)$ can be expressed as $(t > 0)$

$$b^n_{2\ell}(t) = \sum_{k=0}^{\infty} \frac{M_k e^{-t\lambda_k^{(\alpha,\beta)}}}{\mathrm{Vol}(\mathscr{X})} \sum_{j=1}^{\ell} c^\ell_j [\lambda_k^{(\alpha,\beta)}]^j = \frac{1}{\mathrm{Vol}(\mathscr{X})} \mathrm{tr} \left\{ \mathscr{R}_\ell(-\Delta_{\mathscr{X}}) e^{t\Delta_{\mathscr{X}}} \right\}, \tag{2.10}$$

and so in this case one can alternatively express

$$\mathrm{Vol}(\mathscr{X}) b^n_{2\ell}(t) = \sum_{k=0}^{\infty} M_k \left\{ \sum_{j=1}^{\ell} (-1)^j c^\ell_j \frac{d^j}{dt^j} \right\} e^{-t\lambda_k^{(\alpha,\beta)}} = \left\{ \mathscr{R}_\ell \left(-\frac{d}{dt} \right) \right\} \mathrm{tr}\, e^{t\Delta_{\mathscr{X}}}. \tag{2.11}$$

The above analysis nicely underlines the role of the polynomials $\mathscr{R}_\ell$ and the Jacobi coefficients c^ℓ_j in expressing the Maclaurin coefficients $b^n_{2\ell}$ associated with the Schwarz kernel K_Φ of $\Phi(-\Delta_{\mathscr{X}})$. (For related but different results and discussions see also [1–4] as well as [10–15].)

Table 3: Data for the symmetric spaces $\mathbf{P}^n(\mathbb{H})$ and $\mathbf{P}^2(\mathrm{Cay})$

$\mathscr{X}$	$\mathbf{P}^n(\mathbb{H})$	$\mathbf{P}^2(\mathrm{Cay})$
$\lambda_k(\mathscr{X})$	$k(k+2n+1)$	$k(k+11)$
$M_k(\mathscr{X})$	$\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)} \left[\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right]^2$	$6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$
$\mathrm{Vol}(\mathscr{X})$	$\frac{(4\pi)^{2n}}{\Gamma(2n+2)}$	$\frac{3!}{11!}(4\pi)^8$

3 Explicit calculations of the first coefficients $(c^\ell_j : 1 \le j \le \ell)$ and the polynomials $\mathscr{R}_\ell$

The proof of Theorem 2.1 does not reveal in explicit form the Jacobi coefficients. The aim here is to compute the first few in the sequence explicitly. For the sake of brevity hereafter we set $y = P_k^{(\alpha,\beta)}$. Notice that the scalars A^ℓ_m below refer to those in the proof of Theorem 2.1.

- ($\ell = 1$) Indeed

$$\frac{d^2}{d\theta^2} y(\cos\theta)\Big|_{\theta=0} = \mathsf{A}_1^1 y'(1) = \mathsf{A}_1^1 \frac{(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{2\Gamma(\alpha+2)(k-1)!} \tag{3.1}$$
$$= \mathsf{A}_1^1 \frac{k(k+\alpha+\beta+1)}{2(\alpha+1)} y(1)$$
$$\Longrightarrow c_1^1(\alpha,\beta) = -\frac{1}{2(\alpha+1)}.$$

- ($\ell = 2$) Here we have

$$\frac{d^4}{d\theta^4} y(\cos\theta)\Big|_{\theta=0} = \mathsf{A}_1^2 y'(1) + \mathsf{A}_2^2 y''(1) \tag{3.2}$$
$$= \mathsf{A}_1^2 \frac{(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{2\Gamma(\alpha+2)(k-1)!}$$
$$+ \mathsf{A}_2^2 \frac{(k+\alpha+\beta+2)\Gamma(k+\alpha+1)}{4\Gamma(\alpha+3)(k-2)!}(k+\alpha+\beta+1).$$

Simplifying further we see that

$$y''(1) = \frac{(k-1)(k+\alpha+\beta+2)}{4(\alpha+1)(\alpha+2)} k(k+\alpha+\beta+1) y(1) \tag{3.3}$$
$$= \left\{ \frac{[k(k+\alpha+\beta+1)]^2}{4(\alpha+1)(\alpha+2)} - \frac{(\alpha+\beta+2)k(k+\alpha+\beta+1)}{4(\alpha+1)(\alpha+2)} \right\} y(1).$$

Hence we obtain

$$\frac{d^4}{d\theta^4} y(\cos\theta)\Big|_{\theta=0} = \left\{ \frac{\mathsf{A}_1^2(2\alpha+4) - \mathsf{A}_2^2(\alpha+\beta+2)}{4(\alpha+1)(\alpha+2)} k(k+\alpha+\beta+1) \right.$$
$$\left. + \frac{\mathsf{A}_2^2 [k(k+\alpha+\beta+1)]^2}{4(\alpha+1)(\alpha+2)} \right\} y(1). \tag{3.4}$$

Clearly basic differentiation gives $\mathsf{A}_1^2 = 1$, $\mathsf{A}_2^2 = 3$ and so we obtain

$$c_1^2(\alpha,\beta) = -\frac{\alpha+3\beta+2}{4(\alpha+1)(\alpha+2)}, \qquad c_2^2(\alpha,\beta) = \frac{3}{4(\alpha+1)(\alpha+2)}. \tag{3.5}$$

- ($\ell = 3$) Here we have

$$\frac{d^6}{d\theta^6} y(\cos\theta)\Big|_{\theta=0} = \mathsf{A}_1^3 y'(1) + \mathsf{A}_2^3 y''(1) + \mathsf{A}_3^3 y'''(1) \tag{3.6}$$
$$= \mathsf{A}_1^3 y'(1) + \mathsf{A}_2^3 y''(1) + \mathsf{A}_3^3 (k+\alpha+\beta+1) \times$$
$$\times \frac{(k+\alpha+\beta+2)(k+\alpha+\beta+3)\Gamma(k+\alpha+1)}{8\Gamma(\alpha+4)(k-3)!}.$$

Further simplification gives

$$\begin{aligned}\frac{y'''(1)}{y(1)} =&\frac{(k-1)(k-2)(k+\alpha+\beta+2)(k+\alpha+\beta+3)}{8(\alpha+1)(\alpha+2)(\alpha+3)}k(k+\alpha+\beta+1)\\ =&\left\{\frac{[k(k+\alpha+\beta+1)]^3}{8(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{(3\alpha+3\beta+8)[k(k+\alpha+\beta+1)]^2}{8(\alpha+1)(\alpha+2)(\alpha+3)}\right.\\ &\left.+\frac{2(\alpha+\beta+3)(\alpha+\beta+2)k(k+\alpha+\beta+1)}{8(\alpha+1)(\alpha+2)(\alpha+3)}\right\}. \end{aligned}\tag{3.7}$$

So it follows that

$$\begin{aligned}&(3.6)=\Big\{c_1^3(\alpha,\beta)k(k+\alpha+\beta+1)+\\ &+c_2^3(\alpha,\beta)[k(k+\alpha+\beta+1)]^2+c_3^3(\alpha,\beta)[k(k+\alpha+\beta+1)]^3\Big\}y(1),\end{aligned}$$

where we have

$$\begin{aligned}c_1^3(\alpha,\beta)=&\frac{4\mathsf{A}_1^3(\alpha+2)(\alpha+3)-2\mathsf{A}_2^3(\alpha+3)(\alpha+\beta+2)}{8(\alpha+1)(\alpha+2)(\alpha+3)}+\\ &+\frac{2\mathsf{A}_3^3(\alpha+\beta+3)(\alpha+\beta+2)}{8(\alpha+1)(\alpha+2)(\alpha+3)},\\ c_2^3(\alpha,\beta)=&\frac{2\mathsf{A}_2^3(\alpha+3)-\mathsf{A}_3^3(3\alpha+3\beta+8)}{8(\alpha+1)(\alpha+2)(\alpha+3)},\\ c_3^3(\alpha,\beta)=&\frac{\mathsf{A}_3^3}{8(\alpha+1)(\alpha+2)(\alpha+3)}.\end{aligned}\tag{3.8}$$

Clearly by a basic differentiation we see that $\mathsf{A}_1^3=-1$, $\mathsf{A}_2^3=\mathsf{A}_3^3=-15$ and thus

$$c_1^3(\alpha,\beta)=-\frac{4\alpha^2+30\alpha\beta+30\beta^2+20\alpha+60\beta+24}{8(\alpha+1)(\alpha+2)(\alpha+3)},\tag{3.9}$$

$$c_2^3(\alpha,\beta)=\frac{15(\alpha+3\beta+2)}{8(\alpha+1)(\alpha+2)(\alpha+3)},\tag{3.10}$$

$$c_3^3(\alpha,\beta)=-\frac{15}{8(\alpha+1)(\alpha+2)(\alpha+3)}.\tag{3.11}$$

- ($\ell=4$) Indeed from (2.2) we have

$$\frac{d^8}{d\theta^8}y(\cos\theta)\Big|_{\theta=0}=\mathsf{A}_1^4y'(1)+\mathsf{A}_2^4y''(1)+\mathsf{A}_3^4y'''(1)+\mathsf{A}_4^4y^{(4)}(1),\tag{3.12}$$

where $y'(1), y''(1)$ and $y'''(1)$ are as above while

$$\begin{aligned}\frac{y^{(4)}(1)}{y(1)} =& k(k+\alpha+\beta+1)(k-1)(k-2)(k-3)\times \\ &\times \frac{(k+\alpha+\beta+2)(k+\alpha+\beta+3)(k+\alpha+\beta+4)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}. \end{aligned} \tag{3.13}$$

Therefore we have

$$\begin{aligned}(3.13) = &\left\{ \frac{[k(k+\alpha+\beta+1)]^4 - 2(3\alpha+3\beta+10)[k(k+\alpha+\beta+1)]^3}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \right. \\ &+ \frac{\left[11\left(\alpha^2+\beta^2\right)+70(\alpha+\beta)+22\alpha\beta+108\right][k(k+\alpha+\beta+1)]^2}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \\ &\left. - \frac{6(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)k(k+\alpha+\beta+1)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \right\}.\end{aligned}$$

It now follows that

$$\begin{aligned}(3.12) = &\Big\{ c_1^4(\alpha,\beta)k(k+\alpha+\beta+1) + c_2^4(\alpha,\beta)[k(k+\alpha+\beta+1)]^2 + \\ &+ c_3^4(\alpha,\beta)[k(k+\alpha+\beta+1)]^3 + c_4^4(\alpha,\beta)[k(k+\alpha+\beta+1)]^4 \Big\} y(1),\end{aligned}$$

where

$$\begin{aligned}c_1^4(\alpha,\beta) =& \frac{8\mathsf{A}_1^4(\alpha+2)(\alpha+3)(\alpha+4) - 4\mathsf{A}_2^4(\alpha+3)(\alpha+4)(\alpha+\beta+2)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} + \\ &+ 4\frac{\mathsf{A}_3^4(\alpha+4)(\alpha+\beta+2)(\alpha+\beta+3)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} - \\ &- \frac{6\mathsf{A}_4^4(\alpha+\beta+2)(\alpha+\beta+3)(\alpha+\beta+4)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}, \\ c_2^4(\alpha,\beta) =& \frac{4\mathsf{A}_2^4(\alpha+3)(\alpha+4) - 2\mathsf{A}_3^4(\alpha+4)(3\alpha+3\beta+8)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} + \\ &+ \frac{\mathsf{A}_4^4\left[11\left(\alpha^2+\beta^2\right)+70(\alpha+\beta)+22\alpha\beta+108\right]}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}, \\ c_3^4(\alpha,\beta) =& \frac{2\mathsf{A}_3^4(\alpha+4) - 2\mathsf{A}_4^4(3\alpha+3\beta+10)}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}, \\ c_4^4(\alpha,\beta) =& \frac{\mathsf{A}_4^4}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}.\end{aligned} \tag{3.14}$$

Further differentiation gives $\mathsf{A}_1^4 = 1$, $\mathsf{A}_2^4 = 63$, $\mathsf{A}_3^4 = 210$, $\mathsf{A}_4^4 = 105$ and we get

$$\begin{aligned} c_1^4(\alpha,\beta) &= -\frac{34\alpha^3 + 462\alpha^2\beta + 1050\alpha\beta^2 + 630\beta^3 + 306\alpha^2}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} + \\ &\quad - \frac{2184\alpha\beta + 2310\beta^2 + 884\alpha + 2604\beta + 816}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}, \end{aligned} \tag{3.15}$$

$$c_2^4(\alpha,\beta) = \frac{147\alpha^2 + 1050\alpha\beta + 1155\beta^2 + 714\alpha + 2310\beta + 924}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}, \tag{3.16}$$

$$c_3^4(\alpha,\beta) = -\frac{210\alpha + 630\beta + 420}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}, \tag{3.17}$$

$$c_4^4(\alpha,\beta) = \frac{105}{16(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}. \tag{3.18}$$

Acknowledgements

The work of ROA was supported by a PhD scholarship from the Nigerian Ministry of Higher Education and Research and is gratefully acknowledged.

References

[1] R.O. Awonusika, A. Taheri, *Harmonic Analysis on Symmetric and Locally Symmetric Spaces*, In preparation, 2017, (pp. 504+iv).

[2] R.O. Awonusika, A. Taheri, *Spectral Invariants on Compact Symmetric Spaces: From Heat Trace to Functional Determinants*, Submitted 2017, (pp. 262+ix).

[3] D. Bakry, I. Gentil, M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren der Mathematischen Wissenschaften, Vol. **348**, Springer, 2008.

[4] M. Berger, P. Gauduchon, E. Mazet, *Le spectre dúne variétè Riemannienne*, Springer, 1971.

[5] I.S. Gradshtejn, I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 2007.

[6] S. Helgason, Eigenspaces of the Laplacian; integral representations and irreducibility, *J. Funct. Anal.*, Vol. **17**, 1974, pp. 328-353.

[7] S. Helgason, *Topics in Harmonic Analysis on Homogeneous Spaces*, Birkhäuser, 1981.

[8] T.H. Koornwinder, The addition formula for Jacobi polynomials: **I** Summary of results, *Indag. Math.*, Vol. **34**, 1974, pp. 188-191.

[9] T.H. Koornwinder, A new proof of a Paley-Wiener type theorem for the Jacobi transform, *Ark. Matematik*, Vol. **13**, 1975, pp. 145-159.

[10] H. McKean, I.M. Singer, Curvature and the eigenvalues of the Laplacian, *J. Diff. Geom.*, Vol. 1, 1967, pp. 4369.

[11] C. Morris, A. Taheri, On Weyl's asymptotics and remainder term for the orthogonal and unitary groups, *J. Fourier Anal. Appl.*, To appear 2017.

[12] B. Osgood, R. Phillips, P. Sarnak, Extremals and determinants of Laplacians, *J. Funct. Anal.*, Vol. **80**, 1988, pp. 148-211.

[13] P. Sarnak, Determinants of Laplacians, *Comm. Math. Phys.*, Vol. **110**, 1987, pp. 113-120.

[14] P. Sarnak, Determinants of Laplacians: Heights and Finiteness, In: *Analysis, et cetera*, 1990, Academic Press, Boston, MA.

[15] R. Seeley, Complex powers of an elliptic operator, In: *Singular Integrals, Proc. Sympos. Pure Math.*, Chicago, **III**, pp. 288-307, Amer. Math. Soc. Providence, R.I., 1966.

[16] A. Taheri, *Function Spaces and Partial Differential Equations*, **I** & **II**, Oxford Lecture Series in Mathematics and Its Applications, Vols **40** & **41**, OUP, 2015.

[17] N.J. Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs, Vol. **22**, AMS, 1968.

[18] G. Warner, *Harmonic Analysis on Semisimple Lie Groups*, Vols **I** & **II**, Springer, 1972.